Write-up 12: Interesting Relationships between Terms in the Fibonacci Sequence

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Many are (and should be) familiar with the famous Fibonacci sequence,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$$

or, defined recursively,

$$F_0 = 1,\ F_1 = 1$$

$$F_{n+1} = F_n + F_{n-1}, n \geq 1$$

One of the most fundamental features of the Fibonacci sequence is the limit of the ratio of its consecutive terms. This limit is represented by

$$L = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$$

Because the sequence is infinite, we could equivalently represent its limit as such:

$$\lim_{n\to\infty} \frac{f_{n+2}}{f_{n+1}}$$

Setting these two expressions for the limit of the ratio of consecutive terms of the sequence equal to each other,

$$L = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \frac{f_{n+2}}{f_{n+1}}$$

$$= \lim_{n\to\infty} \frac{f_n + f_{n+1}}{f_{n+1}}$$

$$= \lim_{n\to\infty} (\frac{f_n}{f_{n+1}} + 1)$$

$$= 1 + \lim_{n\to\infty} \frac{1}{\frac{f_{n+1}}{f_n}}$$

$$= 1 + \frac{1}{L}$$

Hence,

$$L = 1 + \frac{1}{L}$$

$$\Rightarrow L^2 = L + 1$$

$$\Rightarrow L = \frac{1+\sqrt{5}}{2}$$

(Note: $L \neq \frac{1-\sqrt{5}}{2} < 0$ because $f_n > 0 \; \forall \; n$.) This L is the well known ϕ, or the *golden ratio* (approximately 1.61803399...). Using Excel, we can explore this golden ratio and examine other interesting ratios within the Fibonacci sequence.
For instance, what is the ratio between every second terms, i.e. what is

$$\lim_{n\to\infty} \frac{f_{n+2}}{f_n}$$

What about every third term? Fourth, fifth, etc.? Using Excel, we can produce the following spreadsheet (**Figure 1**) with these ratios as n increases.

We generate these values using very simple functions in the first possible row of each column and then dragging that formula throughout the rows of each column; note that, for example, calculating the ratio between third terms requires starting with the fourth value of the Fibonacci sequence.
It is clear that the ratio of every third terms is not ϕ; in fact, each ratio increases as the terms spread farther apart. Intuitively, this is clear from the increasing nature of the Fibonacci sequence. Further, we can notice that the ratio of every second term is $1 + \phi$; can we prove this?

$$\lim_{n\to\infty} \frac{f_{n+2}}{f_n}$$

$$= \lim_{n\to\infty} \frac{f_n + f_{n+1}}{f_n}$$

$$= \lim_{n\to\infty} 1 + \frac{f_{n+1}}{f_n}$$

$$= 1 + \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$$

$$= 1 + \phi$$

	A	B	C	D	E	F
1	0					
2	1					
3	1	1				
4	2	2	2			
5	3	1.5	3	3		
6	5	1.666667	2.5	5	5	
7	8	1.6	2.666667	4	8	8
8	13	1.625	2.6	4.333333	6.5	13
9	21	1.615385	2.625	4.2	7	10.5
10	34	1.619048	2.615385	4.25	6.8	11.33333
11	55	1.617647	2.619048	4.230769	6.875	11
12	89	1.618182	2.617647	4.238095	6.846154	11.125
13	144	1.617978	2.618182	4.235294	6.857143	11.07692
14	233	1.618056	2.617978	4.236364	6.852941	11.09524
15	377	1.618026	2.618056	4.235955	6.854545	11.08824
16	610	1.618037	2.618026	4.236111	6.853933	11.09091
17	987	1.618033	2.618037	4.236052	6.854167	11.08989
18	1597	1.618034	2.618033	4.236074	6.854077	11.09028
19	2584	1.618034	2.618034	4.236066	6.854111	11.09013
20	4181	1.618034	2.618034	4.236069	6.854098	11.09019
21	6765	1.618034	2.618034	4.236068	6.854103	11.09016
22	10946	1.618034	2.618034	4.236068	6.854101	11.09017
23	17711	1.618034	2.618034	4.236068	6.854102	11.09017
24	28657	1.618034	2.618034	4.236068	6.854102	11.09017
25	46368	1.618034	2.618034	4.236068	6.854102	11.09017
26	75025	1.618034	2.618034	4.236068	6.854102	11.09017
27	121393	1.618034	2.618034	4.236068	6.854102	11.09017
28	196418	1.618034	2.618034	4.236068	6.854102	11.09017
29	317811	1.618034	2.618034	4.236068	6.854102	11.09017
30	514229	1.618034	2.618034	4.236068	6.854102	11.09017
31	832040	1.618034	2.618034	4.236068	6.854102	11.09017
32	1346269	1.618034	2.618034	4.236068	6.854102	11.09017
33	2178309	1.618034	2.618034	4.236068	6.854102	11.09017
34	3524578	1.618034	2.618034	4.236068	6.854102	11.09017
35	5702887	1.618034	2.618034	4.236068	6.854102	11.09017
36	9227465	1.618034	2.618034	4.236068	6.854102	11.09017
37	14930352	1.618034	2.618034	4.236068	6.854102	11.09017
38	24157817	1.618034	2.618034	4.236068	6.854102	11.09017
39	39088169	1.618034	2.618034	4.236068	6.854102	11.09017
40	63245986	1.618034	2.618034	4.236068	6.854102	11.09017
41	1.02E+08	1.618034	2.618034	4.236068	6.854102	11.09017

Figure 1: Figure 1: The Fibonacci sequence & ratios of consecutive, second, third, fourth, and fifth terms

We now may have a new interesting sequence developing. Let us create the following notation:

$$F_0^* = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \phi$$

$$F_1^* = \lim_{n\to\infty} \frac{f_{n+2}}{f_n} = 1 + \phi$$

From the Excel spreadsheet, we can estimate what the limit of the ratio of third terms, or

$$F_2^* = \lim_{n\to\infty} \frac{f_{n+3}}{f_n}$$

by calculating the difference between columns C and D for relatively large n.

$$D41 - C41 = 4.236068 - 2.618034 = 1.618034$$

Approximately a difference of ϕ! Therefore, we should have $F_2^* = 1 + 2\phi$. Upon further investigation, we can indeed prove that

$$F_2^* = \lim_{n\to\infty} \frac{f_{n+3}}{f_n}$$

$$= \lim_{n\to\infty} \frac{f_{n+1} + f_{n+2}}{f_n}$$

$$= \lim_{n\to\infty} \frac{f_{n+1}}{f_n} + \lim_{n\to\infty} \frac{f_{n+2}}{f_n}$$

$$= F_0^* + F_1^*$$

$$= \phi + (1 + \phi)$$

$$= 1 + 2\phi$$

Thus far, it appears that the sequence of *limits of ratios* of Fibonacci numbers defined by

$$\{F^*(n)\}_0^\infty$$

$$F_0^* = \phi,\ F_1^* = 1 + \phi$$

behaves identically as the Fibonacci sequence. Further, considering that $F_1^* = 1 + \phi$ and $F_2^* = 2 + \phi$, it appears that our new sequence F_n^* could also be written as a sum of two sequences:

$$F_n^* = F_n + \phi F_{n+1}$$

First, let's examine if the Excel spreadsheet can confirm this for F_3^* and F_4^*.

$$E41 - D41 = 6.854102 - 4.236068 = 2.618034 \approx 1 + \phi$$

$$F41 - E41 = 11.09017 - 6.854102 = 4.236068 \approx 1 + 2\phi$$

Given our previous values of F_n^*, we can now claim

$$F_3^* = 2 + 3\phi = F_3 + \phi F_4.$$

$$F_4^* = 3 + 5\phi = F_4 + \phi F_5$$

Let us now try to prove the following claim, which would be a somewhat intuitive, but rather interesting statement as to the nature of how the Fibonacci sequence works beyond its most basic properties.

Claim: $F_n^* = F_n + \phi F_{n+1}$
Proof: We will use strong mathematical induction.
Base case: $k = 2$
As shown above, $F_2^* = 1 + 2\phi = F_2 + \phi F_3$.
Induction hypothesis: Suppose $F_{m-1}^* = F_{m-1} + \phi F_m$ for all $k \in \{3, 4, ..., k-1\}$. Then,

$$F_k^* = \lim_{n\to\infty} \frac{F_{n+k+1}}{F_n}$$

$$= \lim_{n\to\infty} \frac{F_{n+k-1}}{F_n} + \frac{F_{n+k}}{F_n} \text{ (by def'n of } F_n)$$

$$= F_{k-2}^* + F_{k-1}^* \text{ (by def'n of } F_n^*)$$

$$= (F_{k-2} + \phi F_{k-1}) + (F_{k-1} + \phi F_k \text{ (by induction hypothesis)}$$

$$= (F_{k-2} + F_{k-1}) + \phi(F_{k-1} + F_k)$$

$$= F_k + \phi F_{k+1} \text{ (by def'n of } F_n)$$

Therefore, $F_n^* = F_n + \phi F_{n+1}$ for all $n \in \mathbb{N} \setminus \{1\}$.